A Gateway to All Post Graduate Courses

An MHRD Project under its National Mission on Education throught ICT (NME-ICT)

Subject: Physics

Production of Courseware

-Content for Post Graduate Courses

Paper No. : Classical Mechanics

Module : Calculus of Variations

Development Team

Principal Investigator	Prof. Vinay Gupta, Department of Physics and Astrophysics, University of Delhi, Delhi
Paper Coordinator	Ashok Goyal, Department of Physics and Astrophysics, University of Delhi, Delhi
Content Writer	Ashok Goyal, Department of Physics and Astrophysics, University of Delhi, Delhi
Content Reviewer	

Description of Module	
Subject Name	Physics
Paper Name	Classical Mechanics
Module Name/Title	Calculus of Variations
Module Id	4

Calculus of Variations

Contents:

Learning Objectives :

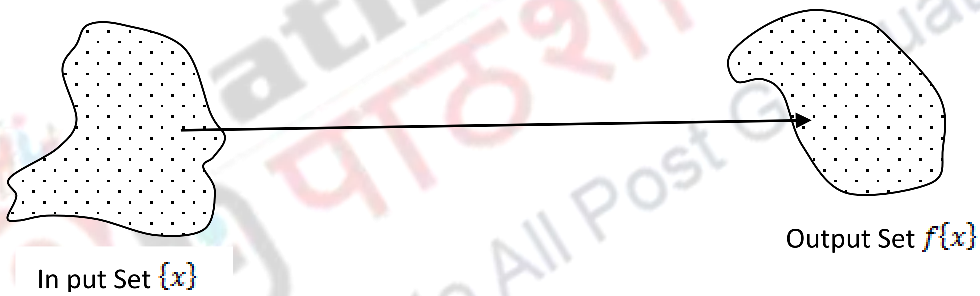
- You will learn about the variational technique to extremize a given functional.
- You will learn many interesting applications of this technique viz equation of the geodesic etc.

1. Introduction:

Newton's equation of motion can be restated in terms of Lagrange's Equations by using **d'Alembert's Principle** as seen in the previous module. The Euler-Lagrange's equation can be derived in an elegant manner by using a **Variational Principle** called **Hamilton's Principle**. The development of Calculation of variation was started by Newton in 1866 and was extended by the Bernoulli brothers, by Euler, Legendre, Langrange, Hamilton and Jacobi to name a few, during the eighteen and early nineteen century.

2. Function and Functional

A function $f(x)$ of x is defined by a rule that maps the input set of numbers $\{x\}$ to an output set of numbers. It may or may not be expressed in terms of an analytic relationship.



A functional $F(y, y', x)$ where x is some function of x and $y' = \frac{dy}{dx}$ and x is an independent variable defines a rule that maps a function on a set of functions on to an output set of numbers, for example

$$I[y] = \int_{x_0}^{x} F(y, y', x)\, dx \qquad (4.1)$$

Is a functional and $F(y, y', x)$ is a simple function that accepts three arguments. For example,we can have

$$F(y, y', x) = ay^2 + by'^2 + Cx^4 \qquad (4.2)$$

The functional $I[y]$ is only one simple specific example of a functional. Every functional need not be in an integral form. For a normal function of n variables $x_1, x_2 \dots \dots \dots x_n$

f $(x_1, x_2 \dots \dots \dots \dots x_n)$ we have by the chain-rule

$$df = \frac{\partial f}{\partial x_1}\delta x_1 + \frac{\partial f}{\partial x_2}\delta x_2 + \cdots \dots \dots$$

$$= \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\delta x_i \tag{4.3}$$

We have a similar form for the variation of a functional.

$$\delta I = \left\{ \int_{x_0}^{x_1} dx \frac{\delta F}{\delta y}(x)\delta y(x) \right\} \tag{4.4}$$

With

$$\frac{\delta F}{\delta y} = \frac{\delta f}{\delta y} - \frac{d}{dx}\frac{\partial F}{dy'} - \frac{d^2}{dx^2}\frac{\partial F}{dy''} \tag{4.5}$$

$$\frac{\delta F(y_1)}{\delta F(y_2)} = \frac{\delta}{\delta F(y_2)}\int dy\, \delta(y - y_1)F(y) \tag{4.6}$$

Hence

$$\frac{\delta F(y_1)}{\delta F(y_2)} = \delta(y_2 - y_1) \tag{4.7}$$

$$\frac{\delta}{\delta F(y)}(\lambda_1 A_1[F] + \lambda_2 A_2[F])$$

$$= \lambda_1 \frac{\delta A_1[F]}{\delta F(y)} + \lambda_2 \frac{\delta A_2[F]}{\delta F(y)} \tag{4.8}$$

$$\frac{\delta(F(y_1)^k)}{\delta F(y_2)} = kF(y_1)^{k-1}\delta(y_2 - y_1) \tag{4.9}$$

and

$$\frac{\delta(F(y_1)^k)}{\delta F(y_2)} = kF(y_1)^{k-1}\delta(y_2 - y_1) \tag{4.9}$$

$$\frac{\delta}{\delta F(y_2)}\int dy_1 \sum_k a_k(y_1)F(y_1)^k$$

$$= \int dy_1\, \delta(y_2 - y_1) \sum_k k\, a_k(y_1) F(y_1)^{k-1} \qquad (4.10)$$

3. Calculus of Variations:

The problem in the calculus of variations is to determine the function $f(x)$ such that the integral x_2.

$$J = \int_{x_1}^{x_2} (fy(x), y'(x); x)\, dx \qquad (4.11)$$

Is an extremum i.e. either a minimum or maximum. In the above equation $y(x)$ is the same function of x $y'(x) = \frac{dy}{dx}$ and x is an independent variable. The $fy\big((x), y'(x); x\big)$ is a functional and is given and the limits are fixed. The function $f(x)$ is then varied until an extremum value of J is found. This means that if a function $y = y(x)$ gives integral J a minimum value, then any neighbouring function however close to $y(x)$ must result in an increase in J.

Let us represent all possible functions $y(x)$ be a parameteric representation $y = y(\alpha, x)$ such that when $\alpha = 0, y = y(0, x) = y(x)$ is the function that extremizes J. We can then write.

$$y = y(\alpha, x) = y(0, x) + \alpha\eta(x) \qquad (2.12)$$

Where $\eta(x)$ is some function of x which has continuous first derivative and which vanishes at the ends points $x_1$and x_2 .

Since $y(\alpha, x) = y(x)$ at the end points of the path.

$$\eta(x_1) = \eta(x_2) = 0 \qquad (2.13)$$

The integral J now becomes a function of α.

For the integral J to have stationary (extremum) value

$$\left.\frac{\partial J}{\partial \alpha}\right|_{\alpha=0} = 0 \qquad (4.15)$$

For all functions $\eta(x)$. Now

$$\frac{\partial J}{\partial \alpha} = \frac{\partial}{\partial \alpha} \int_{x_1}^{x_2} f(y, y'; x)\, dx$$

$$= \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial f}{\partial y'}\frac{\partial^2 y}{\partial \alpha \partial x} \right) dx \tag{4.16}$$

Integrating the second term by parts

$$\int_{x_1}^{x_2} \frac{\partial f}{\partial y'}\frac{d}{dx}\left(\frac{\partial y}{\partial \alpha}\right) \partial x = \frac{\partial f}{\partial y'}\frac{\partial y}{\partial \alpha}\bigg|_{x_1}^{x_2} - \int \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\frac{\partial y}{\partial \alpha} dx \tag{4.17}$$

$$\frac{\partial y}{\partial \alpha} = \eta(x) \quad \therefore \frac{\partial y}{\partial \alpha}\bigg|_{x_1}^{x_2} = \eta(x_2) - \eta(x_1) = 0 \tag{4.18}$$

Thus

$$\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2} \left\{ \frac{\partial f}{\partial y}\frac{\partial y}{\partial \alpha} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\frac{\partial y}{\partial \alpha} \right\} dx$$

$$= \int_{x_1}^{x_2} \left\{ \frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) \right\} \eta(x)\, dx = 0 \tag{4.19}$$

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) - \frac{\partial f}{\partial y} = 0 \tag{4.20}$$

This is Euler's equation and is a necessary condition for J to be extremum. The equation was derived by Euler in the year 1744.

4. Applications:

a) Shortest distance between two points on a plane.

The infinitesimal distance ds between two neighbouring points in the x-y plane is

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \tag{4.21}$$

This distance s between the two points is

$$S = \int_a^b \sqrt{1 + (y')^2}\, dx \tag{4.22}$$

We have to extremize S given the functional

$$f\{y(x), y'(x); x\} = \sqrt{1+(y')^2} \qquad (4.23)$$

For the extremum $f(y, y'; x)$ satisfies the Euler's equation (4.20)

Now

$$\frac{\partial f}{\partial y} = 0 \quad \therefore \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$$

And

$$\frac{\partial f}{\partial y'} = \frac{y'(x)}{\sqrt{1+(y')^2}} = \text{const.} = k \qquad (4.24)$$

$$y'^2 = k^2(1+y'^2) \Longrightarrow y' = \text{constant} = m say$$

Integrating we get

$$y = mx + c \qquad (4.25)$$

Which is the equation of a straight line. Thus we have the well known result, that the shortest distance between two points on a plane lie along the straight line joining the points.

b) Geodesic on a sphere

A geodesic is a line which represents the shortest distance between two points on a surface. The element of length on a sphere of radius r in spherical polar coordinates is given by

$$ds = r(d\theta^2 + \sin^2\theta \, d\theta^2)^{1/2} \qquad (4.26)$$

The distance between two points 1 and 2 on the surface of the sphere is given by

$$S = r\int_1^2 \left\{1 + \sin^2\theta \left(\frac{\partial \phi}{d\theta}\right)^{1/2}\right\} d\theta \qquad (4.27)$$

The shortest distances is obtained when the functional

$$f = \left\{1 + \sin^2\theta \left(\frac{\partial \phi}{d\theta}\right)^2\right\}^{1/2} \qquad (4.28)$$

Satisfies the Euler's equation

$$\frac{d}{d\theta}\left(\frac{\partial f}{\partial \phi'}\right) - \frac{\partial f}{\partial \phi} = 0 \qquad (4.29)$$

Since f does not depend on ϕ

Now

$$\frac{\partial f}{\partial \phi} = 0 \quad and \quad \frac{\partial f}{\partial \phi'} = \frac{\sin^2\theta\,\phi'}{\{1+\sin^2\theta\,\phi'^2\}^{1/2}} \tag{4.30}$$

$$\frac{d}{d\theta}\left(\frac{\partial f}{\partial \phi'}\right) = 0$$

$$\therefore \frac{\partial f}{\partial \phi'} = \text{constant} = C$$

Substituting in (4.30)

$$\frac{\sin^2\theta\,\phi'}{[1+\sin^2\theta\,\phi'^2]^{1/2}} = C \Rightarrow (\sin^4\theta - C^2\sin^2\theta)\phi'^2 = C^2$$

$$\therefore \phi' = \frac{d\phi}{d\theta} = \frac{C}{(\sin^4\theta - C^2\sin^2\theta)^{1/2}} \tag{4.31}$$

$$\phi = C\int \frac{d\theta}{(\sin^4\theta - C^2\sin^2\theta)^{1/2}} + h$$

Let

$$t = c\ \cot\theta, dt = c\,\mathrm{cosec}^2\,d\theta \tag{4.32}$$

$$\therefore \phi = c\int \frac{dt}{(1-C^2-t^2)^{1/2}} + h \tag{4.33}$$

$$\phi = c\sin^{-1}\frac{t}{b} + k \quad \text{where} \quad b^2 = 1-c^2 \tag{4.34}$$

$$b\sin(\phi+k) = k = c\ \cot\theta \tag{4.35}$$

$$\therefore \cos\theta = b\sin\theta\sin\phi\cos k + b\sin\theta\cos\phi\sin k$$

Multiplying by r

$$cr\cos\theta = b\cos k\,(r\sin\theta\sin\phi) + b\sin k(r\sin\theta\sin\phi)$$

$$z = Ax + By \tag{4.36}$$

Where A and B are some constants

This is the equation of a plane that passes through the centre (0,0,0) of the sphere. This plane cuts the surface of the sphere on a circle called the 'great circle'. Thus the shortest distance (geodesic) between two points of a sphere lie on a great circle passing through these points.

c**) The Brachistochrone Problem**

Oone of the classical problems in the calculus of variations is the Brachistrochrone Problem. The problem is to find the path on which a particle moves in the presence of a constant force as to make the time taken by the particle to move from an initial point to a final point minimum. The problem was first solved by Johann Bernolulli in the year 1696. We choose a coordinate system so that the initial point lies at the origin and the force field is directed along the +ve x-axis.

We assume there is no force of friction and the constant force acting on the particle is the gravitational force mg. The total energy of the system during transit is conserved i.e. T+U=cosnt. At the initial point, V is taken to be zero and the particle starts at rest. At any point on the curve.

$$T + U = \frac{1}{2}mv^2 - mgx = 0 \tag{4.37}$$

$$\therefore v = \sqrt{2gx}$$

The time taken by the particle to transit from the initial point (origin) to (x, y) is

$$t = \int_{0,0}^{x,y} \frac{ds}{v} = \int_{0,0}^{x,y} \frac{\sqrt{dx^2 + dy^2}}{v}$$

$$= \frac{1}{\sqrt{2g}} \int_{0,0}^{x,y} \frac{1 + y'^2}{x} dx \tag{4.38}$$

Thus the function to be extremized

$$f = \sqrt{\frac{1 + y'^2}{x}}$$

Satisfies Euler' equation

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) - \frac{\partial f}{\partial y} = 0$$

And since

$$\frac{\partial f}{\partial y} = 0$$

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$$

i.e

$$\frac{\partial f}{\partial y'} = \frac{y'}{\sqrt{(1+y'^2)x}} = \text{constant} = \frac{1}{\sqrt{2a}} \quad \text{say} \tag{4.39}$$

$$y' = \frac{x}{\sqrt{2ax - x^2}}$$

$$\therefore y' = \int \frac{xdx}{\sqrt{2ax - x^2}} \tag{4.40}$$

Let $x = a(1-\cos\theta)$ then $dx = a\sin\theta\, d\theta$ and

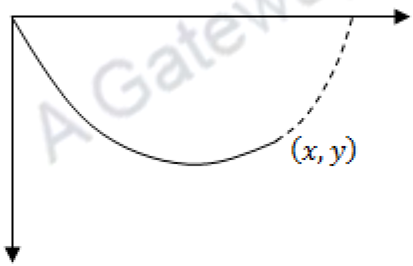
$$y = \int \frac{a^2(1-\cos\theta)\sin\theta\, d\theta}{\{2a^2(1-\cos\theta) - a^2(1-\cos\theta)^2\}^{1/2}}$$

$$y = \int a(1-\cos\theta)d\theta$$

$$y = a(\theta - \mathrm{si.n}\,\theta) + \text{const} \tag{4.41}$$

The parameteric equation of a cycloid passing through the origin is

$$x = a(1-\cos\theta), \quad y = a(1-\sin\theta) \tag{4.42}$$



Thus the curve taken by the particle lies as a ‘cycloid’.

5. Summary:

- The extremization of a functional is achieved if the function $f(y(x)$ satisfies the Euler’s equation

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)$$

- The geodesic on the surface of a sphere between two points lie on a great circle.